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Integrable three-dimensional lattices

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Abstract. Following the work of Zakharov and Shabat, we derive nonlinear equations which are integrable through a discrete Gel'fand–Levitan 'integral' equation in two (resp one, resp zero) continuous and one (resp two, resp three) discrete variables.

1. Introduction

In 1974 Zakharov and Shabat developed a general method to construct Lax pairs, whose associated nonlinear evolution equations (NEEs) are integrable through a Gel'fand–Levitan integral equation.

In this paper we extend the Zakharov–Shabat method to construct nonlinear difference evolution equations (NDEEs) and nonlinear difference difference equations (NDDEs) integrable through a discrete Gel'fand–Levitan equation, by considering three possible cases:

(a) the two time-like variables, which one can associate with the Lax operators, are both continuous;

(b) of the two time-like variables, one is continuous and one is discrete;

(c) both time-like variables are discrete.

In cases (a) and (b), considered in §§ 2 and 3, one obtains NDEEs; in case (c), developed in § 4, one obtains NDDEs.

As examples we exhibit some two-dimensional chiral model equations on a lattice, which reduce, in the scalar case, to the two-dimensional Toda lattice and the discrete analogue of the two-dimensional three-wave equation, both with their Bäcklund transformations (BTs); moreover we display some new integrable three-dimensional lattice equations.

2. The discretised Zakharov–Shabat method

Let us consider a linear discrete 'integral' operator $\hat{F}$ acting on the vector function $\psi(\psi_1(j)\ldots\psi_N(j))$ of the discrete variable j:

$$\hat{F}\psi(n)=\sum_{j=-\infty}^{+\infty}F(n,j)\psi(j).$$

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Here ψ and the $N \times N$ matrix function $F(n, j)$ may depend, in addition to the discrete arguments, parametrically also on other variables continuous or discrete, such as t, y, d, h, etc. We require the following condition for $F(n, j)$:

$$\sup_{n>m} \sum_{j=m}^{+\infty} |F(n,j)| < \infty, \qquad \text{for all } m > -\infty.$$

Let us introduce the Volterra operators $\hat{K}_+$ and $\hat{K}_-$, such that $K_+(n, j) = 0$ for $n \geqslant j$ and $K_-(n, j) = 0$ for $j \geqslant n$, and let us assume that the operator $(1 + \hat{K}_+)$ is invertible. In such a case the representation of $\hat{F}$ in the factorised form

$$(1+\hat{F}) = (1+\hat{K}_+)^{-1}(1+\hat{K}_-)$$

is equivalent to the discrete Gel'fand–Levitan 'integral' equations for $K_+(n, j)$ and $K_-(n, j)$,

$$\begin{aligned} &F(n,j) + K_+(n,j) + \sum_{k=n+1}^{\infty} K_+(n,k)F(k,j) = 0, && j > n, \\ &K_-(n,j) = F(n,j) + \sum_{k=n+1}^{\infty} K_+(n,k)F(k,j), && j < n. \end{aligned} \tag{1}$$

We now introduce two different operators, $\hat{T}_0^{\pm}$, acting on the vector function $\psi(n)$†:

$$\hat{T}_0^{\pm} = \alpha \partial_t + \hat{L}_{0j}^{\pm}, \tag{2}$$

$$\hat{L}_{0j}^{\pm} = l_j^{\pm}(E^{\pm})^j, \tag{3}$$

where j is a positive integer number, α is a scalar constant and l_j is a constant matrix.

Following Zakharov and Shabat (1974) we can prove that, if $\hat{F}$ commutes with $\hat{T}_0^{\pm}$, i.e.‡

$$[\hat{F}, \hat{T}_0^{\pm}] = 0,$$

then we can construct an operator $\hat{T}^{\pm}$ by requiring that the operator equation

$$\hat{T}^{\pm}(1+\hat{K}_{\pm}) = (1+\hat{K}_{\pm})\hat{T}_0^{\pm} \tag{4}$$

be multiplicative.

$$\hat{T}^{\pm} = \alpha \partial_t + \hat{L}_j^{\pm}, \tag{5}$$

$$\begin{aligned} &\hat{L}_j^{+} = l_j^{+}(E^{+})^j + \sum_{k=0}^{j} u_k^{(j)}(n,t)(E^{+})^k, \\ &\hat{L}_j^{-} = l_j^{-}(E^{-})^j + \sum_{k=1}^{j} v_k^{(j)}(n,t)(E^{-})^{j-k}, \end{aligned} \tag{6}$$

where $u_k^{(j)}(n, t)$ and $v_k^{(j)}(n, t)$ are obtained just by requiring that equation (4) be a

† By ∂_z^j we mean the jth partial derivative with respect to z, i.e. $\partial^j/\partial z^j$, and by $E^{\pm}$ the shift operator, such that $E^{\pm}\psi(n) = \psi(n \pm 1)$; in the following we shall also use the notation L_z, by which we mean $(\partial L/\partial z)$.

‡ By $[A, B]$ one means $AB - BA$.

multiplicative operator equation; for instance we have

$$
\begin{aligned}
&u_0^{(j)}(n,t)=0,\\
&u_1^{(j)}(n,t)=-\alpha\,\partial_t K_+(n,n+1),\\
&u_2^{(j)}(n,t)=-\alpha\,\partial_t K_+(n,n+2)-u_1^{(j)}(n,t)K_+(n+1,n+2),\\
&\dots
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&v_1^{(j)}(n,t)=K_+(n,n+1)l_j^- - l_j^- K_+(n-j,n+1-j),\\
&v_2^{(j)}(n,t)=K_+(n,n+2)l_j^- - l_j^- K_+(n-j,n+2-j)-v_1^{(j)}(n,t)K_+(n-j+1,n-j+2),\\
&\dots .
\end{aligned}
\tag{8}
$$

From equation (4), together with the definition of the 'potentials' $u_k^{(j)}(n,t)$ and $v_k^{(j)}(n,t)$, we obtain the evolution equation for $K_+(n,j;t)$

$$\alpha\,\partial_t K_+(n,s)+\hat{L}_j^{\mp}K_+(n,s)-K_+(n,s\mp j)l_j^{\pm}=0, \qquad s>n+j. \tag{9}$$

We can combine the results written in formulae (2), (3), (5), (6), (7) and (8) in the general operator $\hat{T}_0$,

$$\hat{T}_0=\alpha\,\partial_t+\sum_{j=0}^{N^+}\hat{L}_{0j}^+ +\sum_{j=1}^{N^-}\hat{L}_{0j}^- =\alpha\,\partial_t+\hat{\mathscr{L}}_0,$$

where now the potentials $v_k^{(j)}(n,t)$ have the following form†:

$$v_k^{(j)}(n,t)=K_+(n,n+k)l_j^- - l_j^- K_+(n-j,n+k-j)-\sum_{i=1}^{k-1}v_i^{(j)}(n,t)K_+(n-j+i,n-j+k),$$

and, defining $U_k(n,t)=\Sigma_{r=k}^{N^+}u_k^{(r)}(n,t)$, for $U_k(n,t)$ we have

$$
\begin{aligned}
U_0(n,t)&=0,\\
U_k(n,t)&=-\alpha\,\partial_t K_+(n,n+k)-\sum_{s=0}^{k-1}U_s(n,t)K_+(n+s,n+k)+\sum_{s=0}^{k-1}(K_+(n,n+k-s)l_s^+\\
&\quad -l_s^+K_+(n+s,n+k))-\sum_{r=0}^{N^--1}\sum_{s=1}^{N^--r}v_s^{(s+r)}(n,t)K_+(n-r,n+k)\\
&\quad +\sum_{s=1}^{N^-}(K_+(n,n+s+k)l_s^- - l_s^- K_+(n-s,n+k)), \qquad k\geqslant 1.
\end{aligned}
$$

Equation (9) becomes

$$\alpha\,\partial_t K_+(n,s)+\hat{\mathscr{L}}K_+(n,s)-\sum_{j=0}^{N^+}K_+(n,s-j)l_j^+ -\sum_{j=1}^{N^-}K_+(n,s+j)l_j^- =0, \qquad s>n+N^+.$$

It is easy to prove that if the operator $\hat{F}$ satisfies simultaneously the two equations

$$
\begin{aligned}
&[\hat{T}_0^{(1)},\hat{F}]=0, \qquad [\hat{T}_0^{(2)},\hat{F}]=0,\\
&\hat{T}_0^{(1)}=\alpha\,\partial_t+\hat{\mathscr{L}}_0^{(1)}, \qquad \hat{T}_0^{(2)}=\beta\,\partial_y+\hat{\mathscr{L}}_0^{(2)},\\
&\hat{\mathscr{L}}_0^{(j)}=\sum_{k=0}^{N_j^+}l_k^{+(j)}(E^+)^k+\sum_{k=1}^{N_j^-}l_k^{-(j)}(E^-)^k, \qquad j=1,2,
\end{aligned}
$$

† Hereafter the summation is equal to zero if the upper limit is lower than the lower limit.

with $[\hat{\mathscr{L}}_0^{(1)}, \hat{\mathscr{L}}_0^{(2)}]=0$, then the 'dressed' operators $\hat{\mathscr{L}}^{(1)}$ and $\hat{\mathscr{L}}^{(2)}$ satisfy the equation

$$\beta\hat{\mathscr{L}}_y^{(1)}-\alpha\hat{\mathscr{L}}_t^{(2)}=[\hat{\mathscr{L}}^{(1)}, \hat{\mathscr{L}}^{(2)}]. \tag{10}$$

We apply the formalism derived so far to construct, as examples, the Lax pairs corresponding to a set of different non-Abelian Toda lattices, to the Volterra equation and to the discrete two-dimensional three-wave equation.

Setting

$$\hat{\mathscr{L}}^{(1)}=\hat{L}_1^-=l_1E^-+\alpha G^{-1}(n)G_t(n), \qquad \hat{\mathscr{L}}^{(2)}=\hat{L}_1^+=G^{-1}(n)G(n+1)E^+,$$

where $G(n)$ is an $N\times N$ matrix function, we obtain

$$\alpha\beta\,\partial_y[G^{-1}(n)G_t(n)]=l_1G^{-1}(n-1)G(n)-G^{-1}(n)G(n+1)l_1 \tag{11}$$

which becomes in the scalar case, defining $G(n)=\exp[X(n)]$,

$$\delta\beta X_{ty}(n)=\exp[X(n)-X(n-1)]-\exp[X(n+1)-X(n)], \qquad \delta=\alpha/l_1. \tag{12}$$

Setting

$$\hat{\mathscr{L}}^{(1)}=\hat{L}_1^-=l_1E^-+\alpha G^{-1}(n)G_t(n),$$
$$\hat{\mathscr{L}}^{(2)}=\hat{L}_1^++\hat{L}_1^-=G^{-1}(n)G(n+1)E^++l_1E^-+\alpha G^{-1}(n)G_t(n),$$

we obtain

$$\begin{aligned}\alpha\beta\,\partial_y[G^{-1}(n)G_t(n)]-\alpha^2\,\partial_t[G^{-1}(n)G_t(n)]&\\ =l_1G^{-1}(n-1)G(n)-G^{-1}(n)G(n+1)l_1,&\end{aligned} \tag{13}$$

which becomes in the scalar case

$$\delta\beta X_{ty}(n)-\delta\alpha X_{tt}(n)=\exp[X(n)-X(n-1)]-\exp[X(n+1)-X(n)]. \tag{14}$$

Equations (12) and (14) are two different extensions of the Toda lattice to two dimensions; equations (11) and (13) are the corresponding non-Abelian Toda lattice equations, which can be thought of as two-dimensional extensions of discrete chiral models (Popovicz 1980). By a change of variable, both equation (12) and equation (14) can be reduced to the two-dimensional Toda chain considered by Mikhailov (1979). Equation (13) is the appropriate extension to two dimensions of the non-Abelian Toda lattice considered by Bruschi *et al* (1980).

If we set

$$\begin{aligned}\hat{\mathscr{L}}^{(1)}&=\hat{L}_2^++\hat{L}_0^++\hat{L}_2^-=(1+\gamma)A(n)A(n+1)(E^+)^2+D(n)E^+\\ &\quad+A(n)+A(n+1)+(E^-)^2,\\ \hat{\mathscr{L}}^{(2)}&=\hat{L}_1^++\hat{L}_1^-=A(n)E^++E^-,\end{aligned} \tag{15}$$

where $D(n)=\beta\,\Sigma_{j=-\infty}^{n}(A_y(j)+A_y(j-1))$, γ is a constant and $A(n)$ is an $N\times N$ matrix function, we obtain from equation (15)

$$\beta D_{yy}(n)-\alpha A_{ty}(n)=[\gamma/\beta(1+\gamma)](E^+-1)(D(n-1)A(n)-A(n-1)D(n)),$$

which is a non-Abelian Volterra equation whose continuous limit gives the Kadomtsev–Petviashvili equation (Mikhailov 1979).

The discrete two-dimensional three-wave equation can be obtained with the following definitions†.

† By δ_{ij} we mean the usual Kronecker delta function, such that $\delta_{ij}=0$ for $i\neq j$ and $\delta_{ij}=1$ for $i=j$.

$$\hat{\mathscr{L}}_{ij}^{(1)} = \hat{L}_{1,ij}^{-} + \hat{L}_{0,ij}^{+} = b_i\delta_{ij}E^- + (1-\delta_{ij})(b_j - b_i)Q_{ij}(n) + \delta_{ij}b_i(R_i(n)-1),$$
$$R_i(n) = Q_{ii}(n) - Q_{ii}(n-1), \qquad i, j = 1, 2, 3,$$
$$\hat{\mathscr{L}}_{ij}^{(2)} = \hat{L}_{1,ij}^{-'} + \hat{L}_{0,ij}^{+'} = a_i\delta_{ij}E^- + (1-\delta_{ij})(a_j - a_i)Q_{ij}(n) + \delta_{ij}a_i(R_i(n)-1)$$

where $Q_{ij}(n) = Q_{ji}^*(n)$.

By applying equation (10) we obtain

$$\begin{aligned}
&\beta\,\partial_y[b_jQ_{ij}(n) - b_iQ_{ij}(n-1)] - \alpha\,\partial_t[a_jQ_{ij}(n) - a_iQ_{ij}(n-1)] + s_{ij}(E^+ - 1)Q_{ij}(n-1)\\
&\quad = s_{ij}(R_i(n)Q_{ij}(n) - Q_{ij}(n-1)R_j(n)) + s_{rj}Q_{ir}(n)Q_{rj}(n)\\
&\quad - s_{ij}Q_{ir}(n-1)Q_{rj}(n) \qquad\qquad (16)\\
&\quad + s_{ir}Q_{ir}(n-1)Q_{rj}(n-1), \qquad r \neq i \neq j, \qquad s_{ij} = b_ia_j - b_ja_i,\\
&\quad \beta b_i\ \partial_yR_i(n) - \alpha a_i\ \partial_tR_i(n) = \sum_{r\neq i} s_{ri}(E^+ - 1)Q_{ir}(n-1)Q_{ri}(n-1).
\end{aligned}$$

3. Equations in one continuous and two discrete variables

Let us introduce instead of the time derivative operator ∂_t the shift operator D such that

$$Df(n, d) = f(n, d+1)$$

where the variable d spans the space of the integer numbers. In such a case we introduce a new operator $\hat{T}_0$, and consequently $\hat{T}$, satisfying equation (4) in the following way:

$$\hat{T}_0 = (\gamma + \hat{\mathscr{M}}_0)D, \qquad \hat{\mathscr{M}}_0 = \sum_{j=0}^{N^+} \hat{L}_{0j}^+ + \sum_{j=1}^{N^-} \hat{L}_{0j}^-,$$

$$\hat{T} = (\gamma + \hat{\mathscr{M}})D, \qquad \mathscr{M} = \sum_{j=0}^{N^+} \hat{L}_j^+ + \sum_{j=1}^{N^-} \hat{L}_j^-,$$

where $\hat{L}_{0j}^{\pm}$ are defined by equation (3) and $\hat{L}_j^{\pm}$ by equation (6) with the real variable t substituted by the integer variable d. With this definition of $\hat{T}_0$, the potentials $u_k^{(j)}(n, d)$, $v_k^{(j)}(n, d)$ are related to the solution of the discrete Gel'fand–Levitan 'integral' equation (1) in the following way:

$$\begin{aligned}
v_k^{(j)}(n, d) &= K_+(n, n+k\,; d)l_j^- - l_j^-K_+(n-j, n-j+k\,; d+1)\\
&\quad - \sum_{s=1}^{k-1} v_s^{(j)}(n, d)K_+(n-j+s, n-j+k\,; d+1),
\end{aligned}$$

and, defining $U_k(n, d) = \Sigma_{r=k}^{N^+} u_k^{(r)}(n, d)$, we have for $U_k(n, d)$

$$U_0(n, d) = 0,$$

$$\begin{aligned}
U_k(n, d) &= -\gamma(K_+(n, n+k\,; d+1) - K_+(n, n+k\,; d))\\
&\quad - \sum_{s=0}^{k-1} U_s(n, d)K_+(n+s, n+k\,; d+1)\\
&\quad + \sum_{s=0}^{k+1} (K_+(n, n+k-s\,; d)l_s^+ - l_s^+K_+(n+s, n+k\,; d+1))
\end{aligned}$$

$$-\sum_{r=0}^{N^--1}\sum_{s=1}^{N^--r} v_s^{(s+r)}(n,d)K_+(n-r,n+k;d+1)$$

$$+\sum_{s=1}^{N^-}(K_+(n,n+k+s;d)l_s^- - l_s^-K_+(n-s,n+k;d+1)), \qquad k\geqslant 1.$$

The solution $K_+(n,s;d)$ of the discrete Gel'fand–Levitan integral equation (1) now satisfies the following equation:

$$\gamma(K_+(n,s;d+1)-K_+(n,s;d))+\hat{\mathcal{M}}K_+(n,s;d+1)-\sum_{j=0}^{N^+}K_+(n,s-j;d)l_j^+$$

$$-\sum_{j=1}^{N^-}K_+(n,s+j;d)l_j^- = 0, \qquad s>n+N^+.$$

We can easily prove that if the operator $\hat{F}$ satisfies simultaneously the two equations

$$[\hat{T}_0^{(1)},\hat{F}]=0, \qquad \hat{T}_0^{(1)}=(\gamma+\hat{\mathcal{M}}_0)D, \qquad \hat{\mathcal{M}}_0=\sum_{k=0}^{N_1^+} l_k^{+(1)}(E^+)^k+\sum_{k=1}^{N_1^-} l_k^{-(1)}(E^-)^k,$$

$$[\hat{T}_0^{(2)},\hat{F}]=0, \qquad \hat{T}_0^{(2)}=(\beta\,\partial_y+\hat{\mathcal{L}}_0^{(2)}), \qquad \hat{\mathcal{L}}_0^{(2)}=\sum_{k=0}^{N_2^+} l_k^{+(2)}(E^+)^k+\sum_{k=1}^{N_2^-} l_k^{-(2)}(E^-)^k,$$

with $[\hat{\mathcal{M}}_0,\hat{\mathcal{L}}_0^{(2)}]=0$, then the 'dressed' operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{L}}$ satisfy the equation

$$\gamma(D\hat{\mathcal{L}}^{(2)}-\hat{\mathcal{L}}^{(2)}D)-\beta\hat{\mathcal{M}}_yD=[\hat{\mathcal{L}}^{(2)},\hat{\mathcal{M}}D]. \tag{17}$$

We can rewrite equation (17) in two different ways, depending on which operator $\hat{T}$ we consider as the spectral operator; the choice $\hat{\mathcal{L}}=\hat{T}^{(2)}$ implies

$$\gamma(D\hat{\mathcal{L}}-\hat{\mathcal{L}}D)=[\hat{\mathcal{L}},\hat{\mathcal{M}}D], \tag{18}$$

i.e.

$$\hat{\mathcal{L}}\psi=\lambda\psi, \qquad (\gamma+\hat{\mathcal{M}})D\psi=\psi\mathcal{B}(\lambda),$$

where the matrix $\mathcal{B}(\lambda)$ depends on the asymptotic behaviour of the functions $\psi(n,y,t)$ and $\psi'(n,y,t)=D\psi(n,y,t)$. Equation (18) is the BT (Levi *et al* 1981) for the spectral operator $\hat{\mathcal{L}}$ once we define

$$u_k^{(j)}(n)=u_k^{(j)}(n,d), \qquad u_k^{(j)\prime}(n)=u_k^{(j)}(n,d+1),$$

where $u_k^{(j)}(n)$ may depend parametrically on y and t.

The other choice $\hat{\mathcal{L}}=\hat{T}^{(1)}$ gives

$$\beta\hat{\mathcal{L}}_y=[\hat{\mathcal{L}},\hat{\mathcal{L}}^{(2)}] \tag{19}$$

i.e.

$$\hat{\mathcal{L}}\psi=\lambda\psi, \qquad \beta\psi_y=-\hat{\mathcal{L}}^{(2)}\psi,$$

which is the Lax equation for the spectral operator $\hat{\mathcal{L}}$ when y is a time-like variable.

In the following we give some examples of systems which fall in the first or second class (i.e. formula (18) or (19)), i.e. the class of BTs of the NDEEs given in the previous section or the class of NDEEs in two discrete variables, n and d, and one continuous 'time' variable y.

The BT for equation (11) is obtained by choosing

$$\hat{\mathcal{M}}=l_1E^- -\gamma+\gamma G^{-1}(n)G'(n) \tag{20}$$

and it is

$$\gamma\beta\ \partial_y[G^{-1}(n)G'(n)] = l_1(G'(n-1))^{-1}G'(n) - G^{-1}(n)G(n+1)l_1$$

which reduces in the scalar case to

$$(\gamma\beta/l_1)\ \partial_y[X'(n)-X(n)] = \exp[X(n)-X'(n-1)] - \exp[X(n+1)-X'(n)].$$

The same choice for $\hat{\mathcal{M}}$ given by equation (20) yields the BT of equation (13); we obtain

$$\begin{aligned}
&\gamma(l_1G^{-1}(n-1)G'(n-1) - G^{-1}(n)G'(n)l_1)\\
&\qquad = \alpha[(G'(n-1))^{-1}G'_t(n-1) - G^{-1}(n)G_t(n)l_1],\\
&\alpha\gamma[(G'(n))^{-1}G'_t(n) - G^{-1}(n)G_t(n)] - \beta\gamma\ \partial_y[G^{-1}(n)G'(n)] = G^{-1}(n)G(n+1)l_1\\
&\qquad -l_1G^{-1}(n-1)G'(n) + \alpha\gamma[(G'(n))^{-1}G_t(n)G^{-1}(n)G'(n) - G^{-1}(n)G'_t(n)\\
&\qquad - G^{-1}(n)G_t(n) + (G'(n))^{-1}G'_t(n)],
\end{aligned}$$

which becomes in the scalar case

$$\begin{aligned}
&\alpha\ \partial_t[X'(n-1)-X(n)] = \gamma[\exp[X'(n+1)-X(n-1)] - \exp[X'(n)-X(n)]],\\
&\alpha\gamma\ \partial_t[X'(n)-X(n)] - \beta\gamma\ \partial_y[X'(n)-X(n)]\\
&\qquad = l_1[\exp[X(n+1)-X'(n)] - \exp[X(n)-X'(n-1)]].
\end{aligned}$$

The BT for the discrete two-dimensional three-wave equation (16) is obtained by setting

$$\hat{\mathcal{M}}_{ij} = \delta_{ij}E^- + \delta_{ij}(Q_i(n) - Q'_i(n-1)) + (1-\delta_{ij})(Q_{ij}(n) - Q'_{ij}(n-1))$$

where $Q_i(n) = Q_{ii}(n)$. Defining $\Delta_r = (E_r^+ - 1)$, $r = n, d$, we obtain

$$\begin{aligned}
&a_i(\gamma + Q_i(n) - Q'_i(n-1))\Delta_n(Q'_i(n-1) - Q_i(n-1)) - \beta\ \partial_y[Q_i(n) - Q'_i(n-1)]\\
&\qquad = \sum_{s\neq i}[a_s(|Q_{is}(n)|^2 - |Q'_{is}(n)|^2) + a_i(Q'_{is}(n-1)Q'_{si}(n)\\
&\qquad - Q_{is}(n-1)Q_{si}(n) - \Delta_nQ_{is}(n-1)Q'_{si}(n-1))],\\
&\gamma(a_jQ'_{ij}(n) - a_iQ'_{ij}(n-1) - a_jQ_{ij}(n) + a_jQ_{ij}(n-1)) - \beta\ \partial_y[Q_{ij}(n) - Q'_{ij}(n-1)]\\
&\qquad = a_r(Q_{ir}(n)Q_{rj}(n) - Q'_{ir}(n-1)Q'_{rj}(n-1)) + a_iQ_{ir}(n-1)(Q'_{rj}(n-1) - Q_{rj}(n))\\
&\qquad + a_jQ'_{rj}(n)(Q'_{ir}(n-1) - Q_{ir}(n)) + a_i[Q_{ij}(n-1)(Q'_j(n-1) - Q_j(n))\\
&\qquad + Q_{ij}(n)\Delta_nQ_i(n-1) + Q'_{ij}(n-1)(Q_i(n-1) - Q'_i(n-1))]\\
&\qquad + a_j[Q_{ij}(n)(Q_j(n) - Q'_j(n)) + Q'_{ij}(n-1)\Delta_nQ'_i(n-1)\\
&\qquad + Q'_{ij}(n)(Q'_i(n-1) - Q_i(n))] + (a_j - a_i)(Q_{ij}(n) - Q'_{ij}(n-1)),\\
&\qquad r \neq i \neq j.
\end{aligned}$$

We end this section with some examples of NDEEs in two discrete variables and one continuous time variable y.

By choosing

$$\hat{\mathcal{L}}^{(2)} = G^{-1}(n,d)G_y(n,d) + lE^-, \qquad \hat{\mathcal{M}} = G^{-1}(n,d)G(n+1,d+1)E^+,$$

we obtain

$$\gamma\beta\Delta_d(G^{-1}(n,d)G_y(n,d)) = lG^{-1}(n-1,d)G(n,d+1) - G^{-1}(n,d)G(n+1,d+1)l,$$

which becomes in the scalar case

$$(\gamma\beta/l)\,\partial_y\Delta_d X(n,d) = \exp[X(n,d+1) - X(n-1,d)] - \exp[X(n+1,d+1) - X(n,d)].$$

By choosing

$$\hat{\mathscr{L}}^{(2)} = G^{-1}(n-1,d)G(n,d)E^+,$$

$$\hat{\mathscr{M}} = G^{-1}(n-1,d)G(n,d+1)E^+ + lE^- + \beta F(n,d),$$

where

$$F(n,d) = \int_y^\infty dy'(G^{-1}(n-1,d)G(n,d)l - lG^{-1}(n-2,d+1)G(n-1,d+1)),$$

we obtain

$$\gamma\Delta_d(G^{-1}(n-1,d)G(n,d)) - \beta\,\partial_y[G^{-1}(n-1,d)G(n,d+1)]$$
$$= G^{-1}(n-1,d)G(n,d)F(n+1,d) - F(n,d)G^{-1}(n-1,d+1)G(n,d+1).$$

4. NDDES

We now discretise both the time variable t and the time-like variable y by introducing a new shift operator H such that $Hf(n,d,h) = f(n,d,h+1)$. We thus have that if the operator $\hat{F}$ satisfies simultaneously the two equations

$$[\hat{T}_0^{(1)}, \hat{F}] = 0, \qquad \hat{T}_0^{(1)} = (\alpha + \hat{\mathscr{L}}_0^{(1)})D,$$

$$[\hat{T}_0^{(2)}, \hat{F}] = 0, \qquad \hat{T}_0^{(2)} = (\beta + \hat{\mathscr{L}}_0^{(2)})H,$$

$$\hat{\mathscr{L}}_0^{(j)} = \sum_{k=0}^{N_j^+} l_k^{+(j)}(E^+)^k + \sum_{k=1}^{N_j^-} l_k^{-(j)}(E^-)^k, \qquad j = 1, 2,$$

with $[\hat{\mathscr{L}}_0^{(1)}, \hat{\mathscr{L}}_0^{(2)}] = 0$, then the dressed operators $\hat{\mathscr{L}}^{(1)}$ and $\hat{\mathscr{L}}^{(2)}$ satisfy the equation

$$\alpha(D\hat{\mathscr{L}} - \hat{\mathscr{L}}D) = [\hat{\mathscr{L}}, \hat{\mathscr{L}}^{(1)}D],$$

i.e.

$$\hat{\mathscr{L}}\psi = \lambda\psi, \qquad (\alpha + \hat{\mathscr{L}}^{(1)})D\psi = \psi,$$

in terms of the spectral operator $\hat{\mathscr{L}} = \hat{T}^{(2)}$.

In the following we give two examples of NDDES; let

$$\hat{\mathscr{L}}^{(1)} = lE^- + \alpha G^{-1}(n,d,h)\Delta_d G(n,d,h), \qquad \hat{\mathscr{L}}^{(2)} = G^{-1}(n,d,h)G(n+1,d,h+1)E^+.$$

Then we obtain

$$\alpha\beta\Delta_h(G^{-1}(n,d,h)\Delta_d G(n,d,h))$$
$$= lG^{-1}(n-1,d+1,h)G(n,d+1,h+1) - G^{-1}(n,d,h)G(n+1,d,h+1)l,$$

which is a non-Abelian Toda lattice in three discrete variables. By choosing

$$\hat{\mathscr{L}}^{(1)} = G^{-1}(n,d,h)G(n+1,d+1,h)E^+ + B(n,d,h) + lE^-,$$

$$\hat{\mathscr{L}}^{(2)} = G^{-1}(n,d,h)G(n+1,d,h+1)E^+,$$

where

$$B(n, d, h) = (l/\beta) \sum_{j=-\infty}^{h} (lG^{-1}(n-1, d+1, j)G(n, d+1, j+1) - G^{-1}(n, d, j)G(n+1, d, j+1)l),$$

we obtain

$$\begin{aligned}\alpha\Delta_d G^{-1}(n, d, h)G(n+1, d, h+1) - \beta\Delta_h G^{-1}(n, d, h)G(n+1, d+1, h) \\ = G^{-1}(n, d, h)G(n+1, d, h+1)B(n+1, d, h+1) \\ - B(n, d, h)G^{-1}(n, d+1, h)G(n+1, d+1, h+1).\end{aligned}$$

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